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Some Decomposition of Normal Projective Curvature Tensor I

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Abstract: **In the present paper, a Finsler space F_n for which the normal projective curvature tensor N^i_{jkh} satisfies $N^i_{jkh} = y^i Y_{jkh}$ and $N^i_{jkh} = y_j Y^i_{kh}$, where Y_{jkh} and Y^i_{kh} are non-zero tensor fileds called *decompositions tensor fields*, we study the properties of such decomposition and decomposition of projective curvature tensor.**

Keywords: **a Finsler space, decompositions tensor, decomposition of projective curvature tensor.**

1. INTRODUCTION

Ram Hit [6] studied a Finsler space whose curvature tensor is decomposable in the form $H^i_{jkh} = y^i Y_{jkh}$. P. N. Pandey ([2],[3]), H. D. Pande and T. A. Khan [1], Prateek Mishra, Kaushal Srivastava and S. B. Mishra [4] dialed with the problem decomposability of curvature tensor L. Berwald and E. Cartan, a number of examples of Finsler space whose curvature tensor are not decomposable have been cited, a necessary and sufficient condition for decomposability of curvature tensor has been obtained. F. Y. A. Qasem [5] discussed all possible decomposition normal projective curvature tensor N^i_{jkh} in a Finsler space, i. e. a number of examples of Finsler space whose normal projective curvature tensor N^i_{jkh} is not decomposable have been cited and different results have been obtained.

Let us consider a set of quantities g_{ij} defined by

(1.1) $$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x,y).$$

The tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in i and j. According to Euler's theorem on homogeneous functions, the vector y_i satisfies the following relations

(1.2) $$y_i y^i = F^2,$$

The tensor H^i_{jkh} is called *h-curvature tensor*, we shall call it *h-curvature tensor of Berwald* . It is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices which defined by

$$H^i_{jkh} := \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rk} G^r_j - h/k.$$

The above tensor satisfies the following:

(1.3) $$a)\ H^i_{jkh} y^j = H^i_{kh}$$

and

$$b)\ H^r_{rkh} = H_{kh} - H_{hk}.$$

The deviation tensor H^i_h is positively homogeneous of degree two in y^i. In view of Euler's theorem on homogeneous functions we have the following relations

(1.4) $$a)\ H^i_{kh} y^k = H^i_h,$$

$b)\ H_k = H_{kr}^r$

and

$c)\ H = \frac{1}{n-1} H_r^r,$

where H is a scalar called *curvature scalar*. Since contracting of indices does not change the homogeneity, hence, the curvature vector H_k and the curvature scalar H are also homogeneous of degree one and two in y^i, respectively.

2. NORMAL PROJECTIVE CURVATURE TENSOR

K. Yano [7] defined the normal projective connection Π_{jk}^i by

(2.1) $$\Pi_{jk}^i = G_{jk}^i - \frac{1}{n+1} y^i G_{jkr}^r.$$

The connection coefficients Π_{jk}^i is positively homogeneous of degree zero in y^i 's and symmetric in their lower indices.

P.N. Pandey [3] obtained a relation between the normal projective curvature tensor N_{jkh}^i and Berwald curvature tenser H_{jkh}^i as follows:

(2.3) $$N_{jkh}^i = H_{jkh}^i - \frac{1}{n+1} y^i \dot{\partial}_j H_{rkh}^r.$$

The normal projective curvature tensor N_{jkh}^i is homogeneous of degree zero in y^i.

Contracting the indices i and j in (2.3) and using the fact that the tensor H_{rkh}^r is positively homogeneous of degree zero in y^i, we get

(2.4) $$N_{rkh}^r = H_{rkh}^r.$$

Transvecting (2.3) by y^j and using (1.3a), we get

(2.5) $$N_{jkh}^i y^j = H_{kh}^i.$$

The normal projective curvature tensor N_{jkh}^i is skew-symmetric in its last two lower indices, i.e.

(2.6) $$N_{jkh}^i = -N_{jhk}^i.$$

The normal projective curvature tensor N_{jkh}^i also satisfies the identity

(2.7) $$N_{jkh}^i + N_{khj}^i + N_{kjh}^i = 0.$$

Contracting the indices i and h in (2.3), we get

(2.8) $$N_{jk} = H_{jk} - \frac{1}{n+1} y^i \dot{\partial}_j H_{rki}^r$$

or

(2.9) $$N_{jk} = H_{jk} - \frac{1}{n+1} \{\dot{\partial}_j (H_{rki}^r y^i) - H_{rkj}^r\}.$$

The projective curvature tensor W_{jkh}^i and the normal projective curvature tensor N_{jkh}^i are connected by

(2.10) $$W_{jkh}^i = N_{jkh}^i + (\delta_k^i M_{hj} - M_{kh}\delta_j^i - k/h)\,,$$

where

(2.11) $$M_{kh} := -\frac{1}{n^2-1}(nN_{kh} + N_{hk})$$

and

(2.12) $$N_{jk} := N_{jkr}^r.$$

The tensor W_{jkh}^i satisfies the following identities

(2.13) a) $W^i_{jkh} y^j = W^i_{kh}$

and

a) $W^i_{kh} y^k = W^i_h$.

3. DECOMPOSITION OF NORMAL PROJECTIVE CURVATURE TENSOR IN FINSLER SPACE

Let us consider the decompositions of the normal projective curvature tensor N^i_{jkh} of a Finsler space is of the type (1,3) as follows :

(3.1) $N^i_{jkh} = y^i Y_{\boldsymbol{jkh}}$

and

(3.2) $N^i_{jkh} = y_j Y^i_{kh}$,

where Y_{jkh} and Y^i_{kh} are non-zero tensor fileds called *decompositions tensor fields*.

Further considering the decomposition of the tensor field (3.1) in the form

(3.3) $N^i_{jkh} = y^i y_j Y_{kh}$

or

(3.4) $N^i_{jkh} = y^i y_h Y_{jk}$.

Let us define

(3.5) $y^j \lambda_j = \sigma$,

such λ_j as recurrence vector and σ is decomposition scalar.

In view of (3.1), the identities (2.6) and (2.7), can be written as

(3.6) $Y_{jkh} + Y_{jhk} = 0$

and

(3.7) $Y_{jkh} + Y_{khj} + Y_{hjk} = 0$.

Thus, we conclude

Theorem 3.1. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.1), then the decomposition tensor field Y_{jkh} satisfies the identities (3.6) and (3.7).*

In view of (3.2), the identities (2.6) and (2.7), can be written as

(3.8) $Y^i_{kh} + Y^i_{hk} = 0$

and

(3.9) $y_j Y^i_{kh} + y_k Y^i_{hj} + y_h Y^i_{jk} = 0$.

Thus, we conclude

Theorem 3.2. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.2), then the decomposition tensor field Y^i_{kh} satisfies the identities (3.8) and (3.9).*

Transecting (3.2) by y^j, using (2.5) and (1.2), we get

(3.10) $H^i_{kh} = F^2 Y^i_{kh}$.

Transvecting (3.10) by y^k and using (1.4a),we get

(3.11) $H^i_h = F^2 Y^i_h$,

where $Y^i_{kh} y^k = Y^i_h$.

Contraction of the indices i and h in (3.10) and using (1.4b), we get

(3.12) $H_k = F^2 Y_k$,

where $Y^r_{kr} = Y_k$.

Contraction of the indices i and h in (3.12) and using (1.4c), we get

(3.13) $H = \frac{F^2}{n-1} Y$,

where $Y^r_r = Y$.

Thus, we conclude

Theorem 3.3. *If the normal projective curvature tensor $N^i_{jkh}$of a Finsler space is decomposable in the form (3.2), then the h(v)- torsion tensor field H^i_{kh}, the deviation tensor field H^i_h, the vector field H_k and the scalar H are decomposable in the form (3.10), (3.11), (3.12) and (3.13), respectively.*

Contraction of the indices i and j in (3.2) and using (2.4), we get

(3.14) $H^r_{rkh} = y_r Y^r_{kh}$.

Using (1.3b) in (3.14), we get

(3.15) $Y^r_{kh} = \frac{1}{y_r}(H_{hk} - H_{kh})$.

In view of (3.2), equ. (3.15) can be written as

(3.16) $N^i_{jkh} = \frac{y^i}{y_j}(H_{hk} - H_{kh})$.

Transecting (3.16) by y^j and using (2.5), we get

(3.17) $H^i_{kh} = y^i(H_{hk} - H_{kh})$.

Thus, we conclude

Theorem 3.4. *If the normal projective curvature tensor $N^i_{jkh}$of a Finsler space is decomposable in the form (3.2), then the normal projective curvature tensor N^i_{jkh} and the h(v)- torsion tensor H^i_{kh} are defined by (3.16) and (3.17) respectively, the tensor H^r_{rkh} is decomposable in the form (3.14) and the decomposable tensor field Y^i_{kh} satisfies (3.15).*

Contraction of the indices i and j in (3.4) using (2.4), we get

(3.18) $H^r_{rkh} = y_h y^r Y_{rk}$.

Using (1.3b) in (3.18), we get

(3.19) $Y_{rh} = \frac{1}{y_h y^r}(H_{hk} - H_{kh})$.

In view of (3.4), equ. (3.19) can be written as

(3.20) $N^i_{jkh} = \frac{y^i}{y^j}(H_{hk} - H_{kh})$.

Transecting (3.20) by y^j and using (2.5), we get

(3.21) $H^i_{kh} = y^i(H_{hk} - H_{kh})$.

Thus, we conclude

Theorem 3.5. *If the normal projective curvature tensor $N^i_{jkh}$of a Finsler space is decomposable in the form (3.4), then the normal projective curvature tensor N^i_{jkh} and the h(v)- torsion tensor H^i_{kh} are defined by (3.20) and (3.21) respectively, the tensor H^r_{rkh} is decomposable in the form (3.18) and the decomposable tensor field Y_{kh} satisfies (3.19).*

Contraction of the indices i and h in (3.3) and using (2.12), we get

(3.22) $N_{jk} = y_j y^r Y_{kr}$.

Contraction of the indices i and j in (3.3), using (2.4) and (1.2), we get

(3.23) $H^r_{rkh} = F^2 Y_{kh}$.

In view of (3.22), (3.23), (3.3) and (1.2), equ. (2.8) can be written as

(3.24) $H_{jk} = y_j y^r Y_{kr} + \frac{y^r}{n+1} \dot{\partial}_j F^2 Y_{kr}$.

Thus, we conclude

Theorem 3.6. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.4), then the Ricci tensor N_{jk} is decomposable in the form (3.23) and the Ricci tensor H_{jk} satisfies (3.24).*

Contraction of the indices i and h in (3.4) using (2.12) and (1.2), we get

(3.25) $N_{jk} = F^2 Y_{jk}$.

In view of (3.23), (3.25), (3.4), and (1.2), equ. (2.9) can be written as

(3.26) $H_{jk} = F^2 Y_{jk} + \frac{1}{n+1}(y_j y^r Y_{rk} + F^2 Y_{jk} + y^r F^2 \dot{\partial}_j Y_{rk})$.

Thus, we conclude

Theorem 3.7. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.4), then the Ricci tensor N_{jk} is decomposable in the form (3.25) and Ricci tensor H_{jk} satisfies (3.26).*

Further considering the decomposition of the tensor field Y_{jkh} in the form

(3.27) $Y_{jkh} = \lambda_j Y_{kh}$.

In view of (2.6), (3.3) and (3.27), we get

(4.28) $Y_{kh} = -Y_{hk}$.

Thus, we conclude

Theorem 3.8. *In an NPR- F_n, under the decomposition (3.1) and (3.27) the decomposition tensor field Y_{kh} is skew – symmetric.*

4. DECOMPOSITION OF PROJECTIVE CURVATURE TENSOR IN FINSLER SPACE

Let us consider a Finsler space whose the normal projective curvature tensor N^i_{jkh} is decomposable in the form (3.3).

In view of (3.3), (2.12) and (3.22), equ.(2.11) can be written as

(4.1) $M_{jk} = -\frac{y_j y^r}{n^2-1}(nY_{kr} + Y_{rk})$.

Thus, we conclude

Theorem 4.1. *If the normal projective curvature tensor N^i_{jkh} of a Finsler space is decomposable in the form (3.3) and under the decomposition (3.22), the tensor M_{jk} satisfies the equ. (4.1).*

In view of (3.3) and (4.1), equ. (2.10) can be written as

(4.2) $W^i_{jkh} = y^i y_j Y_{kh} - \frac{y^r}{n^2-1}\{\delta^i_k(nY_{hr} + Y_{rh})y_j - y_h(nY_{kr} + Y_{rk})\delta^i_j - k/h\}$.

Transvecting (4.2) by y^j and y^k, using (2.13a) , (2.13b) and (1.2), we get

(4.3) $W^i_{kh} = y^i F^2 Y_{kh} - \frac{y^r}{n^2-1}\{\delta^i_k(nY_{hr} + Y_{rh})F^2 - y_h(nY_{kr} + Y_{rk})y^i - k/h\}$.

and

(4.4) $$W_h^i = y^i F^2 Y_{kh} y^k - \frac{y^r y^i}{n^2-1}\{(nY_{hr} + Y_{rh})F^2 - y_h(nY_{kr} + Y_{rk})y^k - k/h\}.$$

Thus, we conclude

Theorem 4.2. *If the normal projective curvature tensor N_{jkh}^i of a Finsler space is decomposable in the form (3.3) and under the decomposition (3.22), then the projective curvature tensor $W_{jkh}^{\ i}$, the projective torsion tensor W_{kh}^i and the projective deviation tensor W_h^i satisfy equ.s (4.2), (4.3) and (4.4), respectively.*

Let us consider a Finsler space whose the normal projective curvature tensor N^i_{jkh} is decomposable in the form (3.4).

In view of (3.25), (3.4) and (2.11), equ. (2.11) can be written as

(4.5) $$M_{jk} = -\frac{F^2}{n^2-1}(nY_{jk} + Y_{kj}).$$

Thus, we conclude

Theorem 4.3. *If the normal projective curvature tensor N_{jkh}^i of a Finsler space is decomposable in the form (3.4)and under the decomposition (3.25), the tensor M_{jk} satisfies the relation (4.5).*

In view of (3.4) and (4.5), equ. (2.10) can be written as

(4.6) $$W_{jkh}^i = y^i y_h Y_{jk} - \frac{F^2}{n^2-1}\{\delta_k^i(nY_{hj} + Y_{jh}) - (nY_{kh} + Y_{hk})\delta_j^i - k/h\}.$$

Transvecting (4.6) by y^j and y^k, using (2.13a), (2.13b) and (1.2), we get

(4.7) $$W_{kh}^i = y^i y_h Y_{jk} y^j - \frac{F^2}{n^2-1}\{\delta_k^i(nY_{hj} + Y_{jh})y^j - (nY_{kh} + Y_{hk})y^i - k/h\}$$

and

(4.8) $$W_h^i = y^i y_h Y_{jk} y^j y^k - \frac{F^2 y^i}{n^2-1}\{(nY_{hj} + Y_{jh})y^j - (nY_{kh} + Y_{hk})y^k - k/h\}.$$

Thus, we conclude

Theorem 4.4. *If the normal projective curvature tensor N_{jkh}^i of a Finsler space is decomposable in the form (3.4) and under the decomposition (3.25), the projective curvature tensor W_{jkh}^i, the projective torsion tensor W_{kh}^i and the projective deviation tensor W_h^i satisfy equ.s (4.6), (4.7) and (4.8), respectively.*

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